Mathematical Appendix

The fair value of a portfolio of fed funds loans is the par value.

Let r_t be the one-period fed funds rate and the discount rate used to calculate present value. Let F be the amount invested. In the next period, the portfolio pays off principal and interest (1+ r_t)F. The one period discount factor is 1/(1+r_t) so the present value of the payoff next period and therefore the fair value of the portfolio is (1+ r_t)F./(1+r_t) which equals F. For N periods, assume the interest payment is paid out and the principal is rolled over. The interest payment in period t+k is simply r_{t+k}F. The discount factor for period t+k is $(1+r_t)(1+r_{t+1})\ldots(1+r_{t+k}) = \delta_{t+k}$. The fair value of the portfolio is therefore:

$$E_t\left(\frac{r_tF}{(1+r_t)}+\frac{r_{t+1}F}{(1+r_t)(1+r_{t+1})}\cdots\frac{(1+r_{t+N})F}{(1+r_t)(1+r_{t+1})\ldots(1+r_{t+N})}\right)$$
$$= F\,E_t\left(\frac{r_t}{(1+r_t)}+\frac{r_{t+1}}{(1+r_t)(1+r_{t+1})}\cdots\frac{r_{t+N-1}}{(1+r_t)(1+r_{t+1})\ldots(1+r_{t+N-1})}\right.$$
$$\left.+\frac{1}{(1+r_t)(1+r_{t+1})\ldots(1+r_{t+N-1})}\right)$$
$$= F\,E_t\left(\frac{r_t}{(1+r_t)}+\frac{r_{t+1}}{(1+r_t)(1+r_{t+1})}\cdots\frac{r_{t+N-2}}{(1+r_t)(1+r_{t+1})\ldots(1+r_{t+N-2})}\right.$$
$$\left.+\frac{1}{(1+r_t)(1+r_{t+1})\ldots(1+r_{t+N-2})}\right)=F$$

The fair value of a deposit franchise with D deposits, no net inflows or outflows, and deposit rate βr_t is βD

Consider the above proof, but with interest payments βr_t instead of r_t and for a deposit franchise that is infinitely lived. In that case, applying the proof above, the fair value of the deposit franchise is

$$\lim_{N\to\infty} E_t\left(\frac{\beta r_tD}{(1+r_t)}+\frac{\beta r_{t+1}D}{(1+r_t)(1+r_{t+1})}\ldots+\frac{\beta(1+r_{t+N})D}{(1+r_t)(1+r_{t+1})\ldots(1+r_{t+N})}-\frac{\beta D}{(1+r_t)(1+r_{t+1})\ldots(1+r_{t+N})}\right)$$
$$=\beta D-\lim_{N\to\infty}E_t\left(\frac{\beta D}{(1+r_t)(1+r_{t+1})\ldots(1+r_{t+N})}\right)=\beta D$$

Note that the result requires that the interest rate is not zero or negative in equilibrium.